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Solution of Einstein's equations for a line-mass of perfect fluid

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Abstract. A global solution of Einstein's equations for an infinite line-mass of perfect fluid is given. The interior solution depends on two arbitrary constants, and both of these are needed to describe the exterior space–time.

1. Introduction

The exterior field of an infinite line-mass, hereafter called the Levi-Civita (LC) solution, was one of the first problems in general relativity to be solved exactly (Levi-Civita 1919). The field can be interpreted in a Newtonian manner through a logarithmic gravitational potential which occurs in the metric. However, the solution shows three important differences from the Newtonian analogue. First, it appears to contain two genuine arbitrary constants instead of one (Marder 1958). Secondly, the topology conferred on the space–time by the infinite line-mass spearing through it makes the geometry globally non-Euclidean: the ratio of the circumference to radius of large circles centred on the mass is not 2π. Thirdly, there is a parameter m occurring in the logarithmic potential which is clearly connected with the mass per unit length, but with *two* values, namely $m=0$ and $m=\frac{1}{2}$, for which space–time is flat.

Representation of the line-mass by a singularity is unsatisfactory because it leaves open the question whether there exists any real matter which can give rise to the field. For this reason one should represent the line-mass by an interior solution of Einstein's equations, and match this to the LC solution. This has been done previously for certain special interiors (Marder 1958, Raychaudhuri and Som 1962). In this paper I solve the problem for a perfect fluid interior with a certain equation of the state. The interior solution used is one given by Evans (1977) who, however, was not concerned with matching it in detail to the LC exterior.

The most important conclusion of this work is to confirm the result of Marder that the LC solution contains two genuine arbitrary constants, which are determined, through the boundary conditions, by the state of the interior.

In § 2 I give the interior solution in a form suitable to the matching problem which is solved in § 3. A physical interpretation of the global solution is given in § 4 and the paper ends with a brief conclusion.

2. The interior solution

The interior solution is (in a notation somewhat different from that of Evans (1977))

$$ds^2=-dr^2-u^{-1/3}\cos^2 kr\, dz^2-k^{-2}u^{-1/3}\sin^2 kr\, d\phi^2+u^{2/3}\, dt \qquad (2.1)$$

where

$$u = \cos^2 kr + 4q^2 \sin^2 kr, \tag{2.2}$$

and we allow the coordinates to range as follows:

$$0 \leqslant r \leqslant b \qquad -\infty < z < \infty \qquad 0 \leqslant \phi \leqslant 2\pi \qquad -\infty < t < \infty, \tag{2.3}$$

the hypersurfaces $\phi = 0$ and $\phi = 2\pi$ being identified; b, k and q are positive constants. The density and pressure corresponding to (2.1) are

$$\rho = k^2(6\pi u^2)^{-1}(5q^2 + u^2) \tag{2.4}$$

$$p = k^2(6\pi u^2)^{-1}(q^2 - u^2) \tag{2.5}$$

so that the matter present has the equation of state

$$\rho = 5p + \pi^{-1}k^2. \tag{2.6}$$

One can think of this matter as consisting of a gas with equation of state $\rho = 5p$ together with a distribution of dust of density $\pi^{-1}k^2$; this is physically reasonable provided $p \geqslant 0$ for $0 \leqslant r \leqslant b$, i.e.

$$q \geqslant u, \tag{2.7}$$

the equality occurring on the boundary $r = b$. If

$$q > 1 \tag{2.8}$$

(2.7) is satisfied throughout the range (2.3) of r provided b is given by

$$\sin^2 kb = (q-1)(4q^2-1)^{-1}. \tag{2.9}$$

Henceforth we shall assume q satisfies (2.8). The solution then has

$$\rho > 0 \qquad p \geqslant 0$$

with the equality occurring only at the boundary $r = b$.

Conditions also have to be satisfied along the central axis of the distribution $r = 0$. These can be obtained by transforming to local cartesian coordinates

$$x = r\cos\phi \qquad y = r\sin\phi$$

and requiring that g_{ik}, as functions of x and y, tend to Euclidean values and be of some suitable differentiability class, say C^3, as $x \to 0$, $y \to 0$. One can check that these conditions are satisfied for our solution; they also entail the satisfaction of the physical requirement $\partial p/\partial r \to 0$ as $r \to 0$.

We have shown that the metric (2.1), subject to (2.2), (2.3), (2.8) and (2.9) represents a physically reasonable interior space–time. In the next section we shall match it to a vacuum exterior.

3. Matching to the exterior solution

The most general solution of the vacuum equations for a diagonal metric with $g_{rr} = -1$ and the remaining g_{ik} functions of r only is

$$ds^2 = -dr^2 - P^2(r-a)^{2n_1}\,dz^2 - Q^2(r-a)^{2n_2}\,d\phi^2 + S^2(r-a)^{2n_3}\,dt^2 \tag{3.1}$$

where P, Q, S and a are arbitrary real constants, and n_1, n_2, n_3 are real constants satisfying

$$n_1+n_2+n_3=1 \qquad n_1^2+n_2^2+n_3^2=1. \tag{3.2}$$

(3.1) is a simple coordinate transformation of the LC metric for the exterior of a line-mass. In the form given here it may be recognised as a complex transform of the Kasner metric (Misner *et al* 1973).

We can parametrise n_1, n_2, n_3 by

$$\begin{aligned} &n_1=-2m(1-2m)M^{-1} \qquad n_2=(1-2m)M^{-1} \qquad n_3=2mM^{-1} \\ &M=4m^2-2m+1, \end{aligned} \tag{3.3}$$

m being a constant which we take to be positive because it plays approximately the part of the mass per unit coordinate length, as will be shown in the next section. We shall also find that m has to be less than $\frac{1}{4}$. The ranges of the coordinates will be taken as

$$r\geqslant b \qquad -\infty<z<\infty \qquad 0\leqslant\phi\leqslant 2\pi \qquad -\infty<t<\infty. \tag{3.4}$$

The hypersurfaces $\phi=0$ and $\phi=2\pi$ will be identified; this is necessary so that we can match (2.1) and (3.1) (see below).

The metric (3.1) has the curious property of being flat for *two* values of m, namely $m=0$ and $m=\frac{1}{2}$. It would be interesting to understand physically how increasing the mass per unit length sufficiently can render space–time flat. Our interior solution throws no light on this because the greatest value of m attainable is $\frac{1}{4}$. The constants P, Q, S can be removed by a scale change of coordinates, but we keep them as they will be needed in the matching process.

We must match (2.1) and (3.1) on the hypersurface

$$H: r=b \qquad -\infty<z<\infty \qquad 0\leqslant\phi\leqslant 2\pi \qquad -\infty<t<\infty. \tag{3.5}$$

With respect to this hypersurface, (2.1) and (3.1) are in gaussian coordinates and these constitute admissible coordinates in Lichnerowicz's sense. Hence we simply demand that the g_{ik} and their first derivatives be continuous on $r=b$. This leads to six equations:

$$P^2(b-a)^{2n_1}=u_b^{-1/3}\cos^2 kb \tag{3.6}$$

$$Q^2(b-a)^{2n_2}=k^{-2}u_b^{-1/3}\sin^2 kb, \tag{3.7}$$

$$S^2(b-a)^{2n_3}=u_b^{2/3}, \tag{3.8}$$

$$n_1/(b-a)=-\tfrac{2}{3}k(1+2q)\{(q-1)/[q(4q-1)]\}^{1/2} \tag{3.9}$$

$$n_2/(b-a)=\tfrac{1}{3}k(1+2q)\{(4q-1)/[q(q-1)]\}^{1/2} \tag{3.10}$$

$$n_3/(b-a)=\tfrac{2}{3}k[(4q-1)(q-1)/q]^{1/2} \tag{3.11}$$

where u_b is the value of u on the boundary $r=b$, which is in fact equal to q. Dividing (3.9) by (3.10) and inserting the values of n_1 and n_2 from (3.3) we obtain

$$m=(q-1)/(4q-1), \tag{3.12}$$

and using this we find that (3.9)–(3.11) provide only one other distinct relation, namely

$$k(b-n[q(q-1)(4q-1)]^{1/2}(4q^2-2q+1)^{-1}. \tag{3.13}$$

We see that since $k>0$ and the right-hand side of (3.13) is positive, $b-a>0$, and since

$r \geqslant b$, $r-a$ cannot vanish in (3.1). Because of (2.8) and (3.12) we can see that our solution will apply for values of m lying between 0 and $\frac{1}{4}$.

We can summarise by saying that *all the boundary conditions are fulfilled provided the constants satisfy* (2.8), (2.9), (3.6), (3.7), (3.8), (3.12) *and* (3.13), *and the metrics* (2.1) (*with range* (2.3)) *and* (3.1) (*with range* (3.4)) *then give a globally regular solution of Einstein's equations referring to a line-mass of perfect fluid.* The solution has no horizon.

4. Physical interpretation

The solution depends on two arbitrary constants k and q, in terms of which all the other constants may be expressed through the relations referred to. k and q may be thought of as determining the interior matter distribution: k the underlying dust density, and q the central pressure p_0 of the gas, which is given by $p_0 = (6\pi)^{-1}k^2(q^2-1)$.

The exterior metric (3.1) also seems to depend on the two constants k and q, and this may be surprising because the Newtonian gravitational field of a line-mass depends only on the mass per unit length. Can we use coordinate transformations to remove one of these constants? The transformation

$$\bar{r} = r - a \qquad \bar{z} = Pz \qquad \bar{\phi} = Q\phi \qquad \bar{t} = St, \tag{4.1}$$

reduces (3.1) to

$$ds^2 = -d\bar{r}^2 - \bar{r}^{2n_1}\,d\bar{z}^2 - \bar{r}^{2n_2}\,d\bar{\phi}^2 + \bar{r}^{2n_3}\,d\bar{t}^2, \tag{4.2}$$

in which, as we see by (3.3) and (3.12), the only constant appearing is q. Though valid locally, this transformation is globally unacceptable because the new coordinate $\bar{\phi}$ no longer satisfies (3.4). In fact $\bar{\phi}$ satisfies

$$0 \leqslant \bar{\phi} \leqslant 2\pi Q \tag{4.3}$$

so Q (which contains both k and q), though it has disappeared from (4.2), now appears in the range of the coordinates. We conclude that the transformation (4.1) does not remove one of the constants from the exterior space–time. (There is no objection to using (4.1) to bring $g_{\bar{z}\bar{z}}$ and $g_{\bar{t}\bar{t}}$ to the forms shown in (4.2), but Q cannot be removed from $g_{\phi\phi}$ in this way.) A similar conclusion follows if we keep the range of ϕ constant and try to remove the constant k by a transformation of the form

$$H(\bar{r}) = r - a \qquad \bar{z} = Pz \qquad \bar{\phi} = \phi \qquad \bar{t} = St,$$

where H is an arbitrary function.

One can check that, as $r \to \infty$, the physical components of the Riemann tensor (obtained by resolving the coordinate components along the legs of a suitable orthonormal tetrad) tend to zero, so the space–time is locally flat at spatial infinity. Globally, however, the three-spaces $t =$ constant are not Euclidean at infinity, as one can see by considering the ratio of the proper circumference to the proper radius of a circle in the two-space $t =$ constant, $z =$ constant:

$$\text{circumference/radius} = 2\pi Q(r-a)^{n_2}/r.$$

Since $n_2 < 1$ for $m \neq 0$ this ratio tends to zero as $r \to \infty$ and the space closes up.

We shall now show that m is approximately the active gravitational mass per unit coordinate length, m_G. We can calculate the exact value of m_G by using Whittaker's (1935) expression

$$m_G = \int_0^b \int_0^1 \int_0^{2\pi} (\rho + 3p)(-g)^{1/2}\, dr\, dz\, d\phi$$

where g is the four-dimensional determinant of the interior metric (2.1). It turns out that

$$m_G = m(1-4m^2)^{-1},$$

so that for the range of our solution, namely $0 < m < \frac{1}{4}$, $m \sim m_G$.

5. Conclusion

We have obtained a global solution of Einstein's equations for an infinite line-mass made of perfect fluid with a particular equation of state depending on two arbitrary constants. It turns out that both these constants are also needed to describe the exterior field. This feature, not present in the corresponding Newtonian case, is connected with the topology of the space resulting from the presence of the infinite line-mass. This topology is also responsible for the property that, although the space–time tends to local flatness at infinity, it is not globally flat.

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